The method of descent

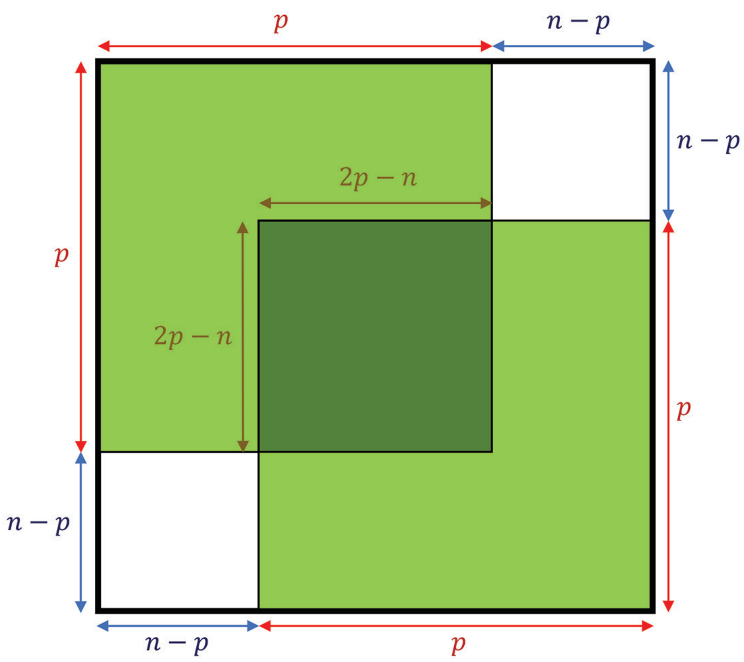
ALEXANDER FARRUGIA

Imagine a square room, of dimensions *n* metres by *n* metres, where *n* is a whole number. We have two identical square carpets, both having dimensions *p* metres by *p* metres, where *p* is also a whole number. The area of the room is twice the area of one of the carpets. We ask the question: what are the *smallest* possible values for *n* and *p*, for which this is true?

For example, let us try *n* is seven metres and *p* is five metres. Then the area of one of the carpets would be five times five, or 25 square metres, while the area of the room would be seven times seven, or 49 square metres. This area is *almost* twice 25 square metres, but not quite, so these values of *n* and *p* do not work.

Suppose, though, that the smallest possible values of *n* and *p* such that twice *p* times *p* is equal to *n* times *n* are available, and let the room and carpets have these dimensions.

We place these carpets on the floor of this room so that one corner of each carpet coincides with each of two opposite corners of the room. Since the area of both carpets together is equal to the area of the room, the area of the overlapping carpets must be equal to the area of the uncovered parts of the room. The overlapping part is a square of dimensions (2*p* – *n*) metres by (2*p* – *n*) metres, while the uncovered parts consist of two squares, each of dimensions (*n* – *p*) metres by (*n* – *p*) metres.



Two square carpets in light green placed in a square room. Their overlap is the dark green square in the middle. The total area of the uncovered room (the two small, white squares) is equal to the area of the dark green square, leading to a contradiction.

We have thus discovered that two identical squares, each of dimensions (*n* – *p*) by (*n* – *p*), together have the same area as a square of dimensions (2*p* – *n*) by (2*p* – *n*).

It is clear that (2*p* – *n*) and (*n* – *p*) are both whole numbers that are less than *p* and *n*. This is absurd, because *p* and *n* were assumed to be the *smallest* possible values that solve this problem. We conclude that no such whole number values for *n* and *p* exist.

The above proof argument is called Fermat's method of descent, after Pierre de Fermat (1607 – 1665) who used it to prove several theorems. In this method, one starts from an assumed smallest solution, then demonstrates that an even smaller solution would exist, contradicting the assumption that the starting solution was the smallest. In this article, we have used Fermat's method of descent to actually show that no square with a rational side length can have an area of two square units.

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MYTH DEBUNKED

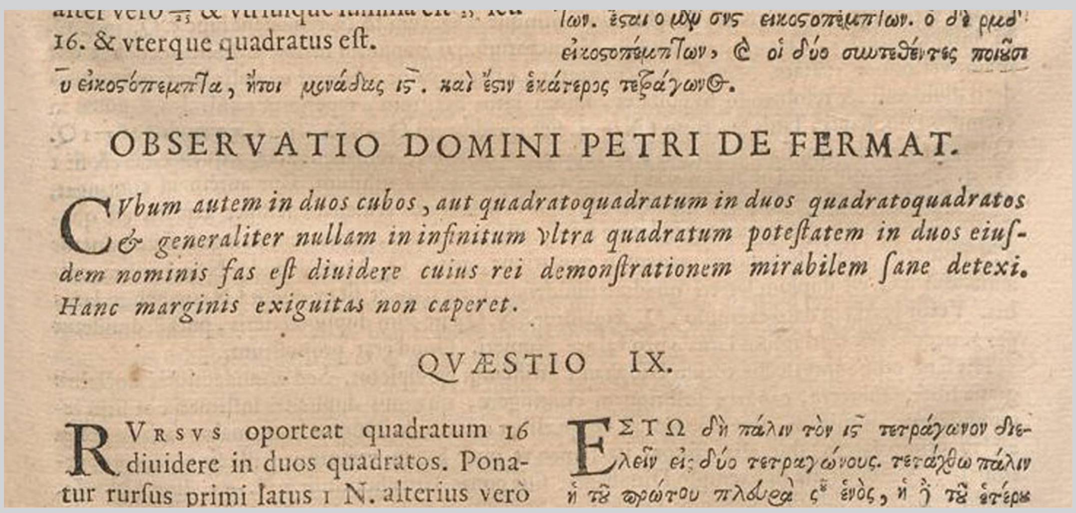
Is truth in mathematics the same as truth in science?

In science, something is true by empirical evidence emerging from observation of physical phenomena and careful devising of experiments. If, within a particular scenario, the same observations are always made, then science claims that, given the same scenario, there is reasonable evidence suggesting that these identical observations will be made in the future. Scientific truth is inductive, not deductive.

For example, Ptolemy's geocentric model of the universe was scientific truth for more than 1500 years. This model could predict solar eclipses with remarkable accuracy; the evidence for it being the correct model was very striking. Nowadays, this model has been superseded by a better one that fits our observations even better than the geocentric model. And we expect science to continue to do so, improving its models to get closer and closer to the absolute truth.

Mathematical truth is very different from scientific truth. In mathematics, a statement is true if it is either one of the statements of the underlying axioms (assumed truths), or the statement follows from them using logical inference rules. Mathematical truth is thus deductive, not inductive. A statement in mathematics that has been proven true will never be superseded, or suddenly asserted to be false. Assuming the underlying set of axioms and inference rules, mathematics uncovers absolute truths.

PHOTO OF THE WEEK


16. & vterque quadratus est.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò

A closeup of page 61 of the 1670 edition of Diophantus' *Arithmetica*, published after Fermat's death by his son Clément-Samuel, who included his father's commentary. This photo focuses on Fermat's comment which would famously become Fermat's Last Theorem. It translates to: "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain." PHOTO: HTTPS://EN.WIKIPEDIA.ORG/WIKI/FERMAT'S_LAST_THEOREM

DID YOU KNOW?

- Pierre de Fermat was a lawyer by profession; doing mathematics was his hobby. His mathematics was imparted by letters, in which often his results were stated with scant proof.
- Fermat is famous for the assertion he made in 1637, nowadays called Fermat's Last Theorem, stating that if *n* is a whole number, the sum of *n*th powers of two whole numbers cannot be equal to the *n*th power of a whole number except when *n* is 1 or 2. He had written this result on a copy of Diophantus' book *Arithmetica* and cheekily added that he had a remarkable proof of the result but the proof was too large to fit in the margin of the book. The result was only proved 358 years later, in 1995, by Andrew Wiles.

For more trivia, see: www.um.edu.mt/think.

SOUND BITES

- In a control system, a network of agents exchange signals with each other to produce an output from an input received by an external agent. A controllable graph is such a system where the input of the external agent, which communicates with the agents with equal sensitivity, is always able to produce a predetermined output. In the paper *Alexander Farrugia, 'On Strongly Asymmetric and Controllable Primitive Graphs', Discrete Applied Mathematics, 211 (2016), 58-67*, Fermat's method of descent was used to demonstrate that one can remove agents from any controllable graph to end up with four agents connected by a path.

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